



TITLE:

On the transfer map for the Hochschild cohomology of Frobenius algebras (Cohomology Theory of Finite Groups and Related Topics)

AUTHOR(S):

Sanada, Katsunori

CITATION:

Sanada, Katsunori. On the transfer map for the Hochschild cohomology of Frobenius algebras (Cohomology Theory of Finite Groups and Related Topics). 数理解析研究所講究録 2008, 1581: 28-33

ISSUE DATE:

2008-02

URL:

<http://hdl.handle.net/2433/81435>

RIGHT:

On the transfer map for the Hochschild cohomology of Frobenius algebras

Katsunori Sanada

Department of Mathematics, Tokyo University of Science

眞田 克典 (東京理科大学理学部数学教室)

1 Introduction

We describe a transfer map between the complete Hochschild cohomologies of Frobenius algebras Λ and Γ . For a Frobenius algebra Λ over a commutative ring R which is finitely generated projective R -module, we can define a complete Hochschild cohomology $H^i(\Lambda, M)$ with a coefficient Λ -bimodule M (see [Na]). If, in addition, we assume that Γ is a Frobenius extension of Λ , then Γ is a Frobenius R -algebra. Under this assumption, we can define $\text{Res} : H^r(\Gamma, {}_\Gamma M_\Gamma) \rightarrow H^r(\Lambda, M)$ and $\text{Cor} : H^r(\Lambda, {}_\Gamma M_\Gamma) \rightarrow H^r(\Gamma, M)$. Res for $r \geq 0$ and Cor for $r \leq -1$ are defined naturally, and, particularly for Frobenius algebras, Res and Cor can also be defined for other integers r , which we may call them the transfer maps (see [S3], [S4], and also [No1], [No2]).

In this summary, we show the explicit description of Res and Cor by means of the standard resolutions of the Frobenius algebras above.

2 Complete Hochschild cohomology of Frobenius algebras

Let R be a commutative ring with identity, Λ an R -algebra which is finitely generated projective as R -module. $\Lambda^e = \Lambda \otimes_R \Lambda^{\text{opp}}$ denotes the enveloping algebra of Λ and $Z\Lambda$ denotes the center of Λ .

If M is a left Λ^e -module (i.e. Λ -bimodule), we define the Hochschild cohomology of Λ with coefficient module M :

$$H^n(\Lambda, M) = \text{Ext}_{\Lambda^e}^n(\Lambda, M) \quad (n \geq 0).$$

It is easily verified that this is a $Z\Lambda$ -module.

We denote $H^n(\Lambda, \Lambda)$ by $HH^n(\Lambda)$ in the following. By the definition, we see that

$$H^0(\Lambda, M) \cong M^\Lambda = \{x \in M \mid ax = xa \text{ for any } a \in \Lambda\}$$

and so we have $HH^0(\Lambda) = Z\Lambda$.

2.1 Standard resolution, cup product

Let $n \geq 0$ be an integer, and we put $X_n = \Lambda \otimes \cdots \otimes \Lambda$ ($n + 2$ -times tensor products over R). Then we have the following Λ^e -projective resolution of Λ which is called the standard resolution of Λ :

$$\begin{aligned} \cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} \Lambda \longrightarrow 0, \\ d_n(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = \sum_{i=0}^n (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}, \\ d_1(x_0 \otimes x_1 \otimes x_2) = x_0 x_1 \otimes x_2 - x_0 \otimes x_1 x_2, \\ d_0(x_0 \otimes x_1) = x_0 x_1 \end{aligned}$$

We can define the cup product $H^i(\Lambda, M) \otimes H^j(\Lambda, N) \xrightarrow{\smile^{i,j}} H^{i+j}(\Lambda, M \otimes_\Lambda N)$, which satisfies the anti-commutativity:

$$\alpha \smile_{i,j} \beta = (-1)^{ij} \beta \smile_{j,i} \alpha \quad \text{for } \alpha \in HH^i(\Lambda), \beta \in HH^j(\Lambda, M).$$

and the cup product $Z\Lambda \otimes H^i(\Lambda, M) \xrightarrow{\smile^{0,i}} H^i(\Lambda, M)$ gives the $Z\Lambda$ -module structure for $H^i(\Lambda, M)$.

Furthermore, we have $HH^i(\Lambda) \otimes HH^j(\Lambda) \xrightarrow{\smile} HH^{i+j}(\Lambda)$, so this makes

$$HH^*(\Lambda) := \bigoplus_{k \geq 0} HH^k(\Lambda)$$

a ring containing $HH^0(\Lambda) = Z\Lambda$ as a subring, which is called the Hochschild cohomology ring of Λ .

2.2 Frobenius extensions and Frobenius algebras

Let Γ/Λ be a Frobenius extension. That is,

$$\begin{aligned} \Gamma &= a_1 \Lambda \oplus \cdots \oplus a_m \Lambda = \Lambda b_1 \oplus \cdots \oplus \Lambda b_m; \\ x a_i &= \sum_{j=1}^m a_j \beta_{ji}(x), \quad b_j x = \sum_{i=1}^m \beta_{ji}(x) b_i \quad (x \in \Gamma, \beta_{ji}(x) \in \Lambda) \end{aligned}$$

and there exist the following isomorphisms:

$$\begin{aligned} \phi_{\Gamma/\Lambda} : \Gamma \Gamma_\Lambda &\xrightarrow{\sim} \text{Hom}_{\Lambda, -}(\Gamma_\Gamma, \Lambda_\Lambda), \quad \phi_{\Gamma/\Lambda}(a_i)(b_j) = \delta_{ij}, \\ \phi'_{\Gamma/\Lambda} : {}_\Lambda \Gamma_\Gamma &\xrightarrow{\sim} \text{Hom}_{-, \Lambda}(\Gamma_\Gamma, {}_\Lambda \Lambda), \quad \phi'_{\Gamma/\Lambda}(b_j)(a_i) = \delta_{ij}. \end{aligned}$$

We set

$$\mu_{\Gamma/\Lambda} = \phi_{\Gamma/\Lambda}(1), \quad N_{\Gamma/\Lambda}(x) = \sum_{i=1}^m a_i x b_i \quad (x \in \Gamma).$$

Then $\mu_{\Gamma/\Lambda} : \Gamma \rightarrow \Lambda$ is a two-sided Λ -module homomorphism and

$$x = \sum_{i=1}^m \mu_{\Gamma/\Lambda}(xa_i)b_i = \sum_{j=1}^m a_j \mu_{\Gamma/\Lambda}(b_j x) \quad (x \in \Gamma).$$

Furthermore, let R be a commutative ring and Λ a Frobenius R -algebra which is finitely generated free R -module:

$$\begin{aligned} \Lambda &= u_1 R \oplus \cdots \oplus u_n R = Rv_i \oplus \cdots \oplus Rv_n; \\ yu_i &= \sum_{j=1}^n u_j \alpha_{ji}(y), \quad v_j y = \sum_{i=1}^n \alpha_{ji}(y)v_i \quad (y \in \Lambda, \alpha_{ji}(y) \in R), \\ \phi_\Lambda : {}_\Lambda \Lambda &\xrightarrow{\sim} \text{Hom}_R(\Lambda_\Lambda, R), \quad \phi_\Lambda(u_i)(v_j) = \delta_{ij}. \end{aligned}$$

We set

$$\begin{aligned} \mu_\Lambda &= \phi_\Lambda(1), \quad N_\Lambda(y) = \sum_{i=1}^n u_i y v_i, \\ y^{\tau'} &= \sum_{i=1}^n \mu_\Lambda(u_i y) v_i \quad (\text{Nakayama automorphism of } \Lambda/R). \end{aligned}$$

Then Γ is a Frobenius R -algebra of rank mn with R -bases $(a_i u_j), (v_j b_i)$ ($1 \leq i \leq m, 1 \leq j \leq n$):

$$\begin{aligned} xa_i u_j &= \sum_{k=1}^m \sum_{l=1}^n a_k u_l \beta_{ij}(\alpha_{ki}(x)), \\ v_l b_k x &= \sum_{i=1}^m \sum_{j=1}^n \beta_{ij}(\alpha_{ki}(x)) v_j b_i \quad (x \in \Gamma), \\ \phi_\Gamma : {}_\Gamma \Gamma &\xrightarrow{\sim} \text{Hom}_R(\Gamma_\Gamma, R), \quad \phi_\Gamma(a_i u_j)(v_l b_k) = \delta_{(i,j),(k,l)}. \end{aligned}$$

If we set $\mu_\Gamma = \phi_\Gamma(1)$, then

$$x = \sum_{i=1}^m \sum_{j=1}^n \mu_\Gamma(xa_i u_j) v_j b_i = \sum_{k=1}^m \sum_{l=1}^n a_k u_l \mu_\Gamma(v_l b_k x) \quad (x \in \Gamma).$$

We set

$$\begin{aligned} N_\Gamma(x) &:= \sum_{i=1}^m \sum_{j=1}^n a_i u_j x v_j b_i, \\ x^\tau &:= \sum_{i=1}^m \sum_{j=1}^n \mu_\Gamma(a_i u_j x) v_j b_i \quad (x \in \Gamma). \end{aligned}$$

Then we have

$$N_{\Gamma/\Lambda} \circ N_\Lambda = N_\Gamma, \quad \mu_\Lambda \circ \mu_{\Gamma/\Lambda} = \mu_\Gamma, \quad \tau|_\Lambda = \tau'.$$

3 Restriction and corestriction maps

We set

$$\begin{aligned}(X_\Gamma)_p &= \Gamma \otimes_R \cdots \otimes_R \Gamma \quad (p+2 \text{ times tensor products of } \Gamma), \\ (X_\Lambda)_p &= \Lambda \otimes_R \cdots \otimes_R \Lambda \quad (p+2 \text{ times tensor products of } \Lambda),\end{aligned}$$

and we define

$$\begin{aligned}d_p : (X_\Gamma)_p &\longrightarrow (X_\Gamma)_{p-1}, \\ d_p(x_0 \otimes x_1 \otimes \cdots \otimes x_p \otimes x_{p+1}) \\ &= x_0 x_1 \otimes \cdots \otimes x_p \otimes x_{p+1} \\ &\quad + \sum_{i=1}^{p-1} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{p+1} + (-1)^p x_0 \otimes \cdots \otimes x_{p-1} \otimes x_p x_{p+1}.\end{aligned}$$

Then we have the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longleftarrow & \text{Hom}_{\Gamma^e}((X_\Gamma)_1, M) & \xleftarrow{d_1^*} & M & \xleftarrow{N_\Gamma} & M & \xleftarrow{d_1 \otimes \iota} & (X_\Gamma)_1^T \otimes_{\Gamma^e} M & \longleftarrow & \cdots \\ & & \downarrow \text{res}^1 & & \downarrow \text{res}^0 & & \downarrow \text{res}_0 & & \downarrow \text{res}_1 & & \\ \cdots & \longleftarrow & \text{Hom}_{\Lambda^e}((X_\Lambda)_1, M) & \xleftarrow{d_1^*} & M & \xleftarrow{N_\Lambda} & M & \xleftarrow{d_1 \otimes \iota} & (X_\Lambda)_1^{T'} \otimes_{\Lambda^e} M & \longleftarrow & \cdots \end{array}$$

Here, $(X_\Gamma)_p^T$ is defined by $w(x \otimes y^{opp}) = y^{\tau^{-1}} w x$ for $w \in (X_\Gamma)_p$, $x \otimes y^{opp} \in \Gamma^e$, and

$$\begin{aligned}\text{res}^0 : M &\rightarrow M, x \mapsto x, \quad \text{res}_0 : M \rightarrow M, x \mapsto \sum_{i=1}^m b_i x a_i^\tau, \\ \text{res}_q : (X_\Gamma)_q^T \otimes_{\Gamma^e} M &\rightarrow (X_\Lambda)_q^{T'} \otimes_{\Lambda^e} M, 1 \otimes y_1 \otimes \cdots \otimes y_q \otimes 1 \otimes_{\Gamma^e} x \mapsto \\ &\sum_{i_1, \dots, i_{q+1}=1}^m 1 \otimes \mu_{\Gamma/\Lambda}(b_{i_1} y_1 a_{i_2}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_q} y_q a_{i_{q+1}}) \otimes 1 \otimes_{\Lambda^e} b_{i_{q+1}} x a_{i_1}^\tau,\end{aligned}$$

res^p ($p \geq 1$) is defined to be a natural homomorphism induced by $(X_\Lambda)_p \rightarrow (X_\Gamma)_p$. Then we have

$$\text{Res}^r : H^r(\Gamma, {}_\Gamma M_\Gamma) \rightarrow H^r(\Lambda, M) \quad (r \in \mathbb{Z}).$$

Here, $H^r(\Gamma, -)$ and $H^r(\Lambda, -)$ denotes the complete Hochschild cohomology of Γ and Λ , respectively, and these are obtained by the horizontal sequences.

On the other hand, we have the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longleftarrow & \text{Hom}_{\Gamma^e}((X_\Gamma)_1, M) & \xleftarrow{d_1^*} & M & \xleftarrow{N_\Gamma} & M & \xleftarrow{d_1 \otimes \iota} & (X_\Gamma)_1^T \otimes_{\Gamma^e} M & \longleftarrow & \cdots \\ & & \uparrow \text{cor}^1 & & \uparrow \text{cor}^0 & & \uparrow \text{cor}_0 & & \uparrow \text{cor}_1 & & \\ \cdots & \longleftarrow & \text{Hom}_{\Lambda^e}((X_\Lambda)_1, M) & \xleftarrow{d_1^*} & M & \xleftarrow{N_\Lambda} & M & \xleftarrow{d_1 \otimes \iota} & (X_\Lambda)_1^{T'} \otimes_{\Lambda^e} M & \longleftarrow & \cdots \end{array}$$

Here,

$$\begin{aligned}
& \text{cor}_0 : M \rightarrow M, x \mapsto x, \quad \text{cor}^0 : M \rightarrow M, x \mapsto N_{\Gamma/\Lambda}(x), \\
& \text{cor}^p : \text{Hom}_{\Lambda^e}((X_\Lambda)_p, M) \rightarrow \text{Hom}_{\Gamma^e}((X_\Gamma)_p, M), \\
& \text{cor}^p(g)(y_0 \otimes y_1 \otimes \cdots \otimes y_p \otimes y_{p+1}) \\
& = \sum_{i_1, \dots, i_{p+1}=1}^m y_0 a_{i_1} g(1 \otimes \mu_{\Gamma/\Lambda}(b_{i_1} y_1 a_{i_2}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_p} y_p a_{i_{p+1}}) \otimes 1) b_{i_{p+1}} y_{p+1},
\end{aligned}$$

and cor_q ($q \geq 1$) is defined to be a natural homomorphism induced by $(X_\Lambda)_q \rightarrow (X_\Gamma)_q$. Then we have

$$\text{Cor}^r : H^r(\Lambda, {}_\Gamma M_\Gamma) \rightarrow H^r(\Gamma, M) \quad (r \in \mathbb{Z}).$$

Proposition We have following fundamental properties for Res and Cor.

(1) Given $f : {}_\Gamma M \rightarrow {}_\Gamma N$, we have

$$\begin{aligned}
f^* \text{Res}^r &= \text{Res}^r f^* : H^r(\Gamma, M) \rightarrow H^r(\Lambda, N), \\
f^* \text{Cor}^r &= \text{Cor}^r f^* : H^r(\Lambda, M) \rightarrow H^r(\Gamma, N).
\end{aligned}$$

(2) Given a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of Γ^e -modules, we have

$$\begin{aligned}
\partial \text{Res}^r &= \text{Res}^{r+1} \partial : H^r(\Gamma, N) \rightarrow H^{r+1}(\Lambda, L), \\
\partial \text{Cor}^r &= \text{Cor}^{r+1} \partial : H^r(\Lambda, N) \rightarrow H^{r+1}(\Gamma, L).
\end{aligned}$$

(3) Given a Γ^e -module M , we have

$$\text{Cor}^r \text{Res}^r(w) = N_{\Gamma/\Lambda}(1)w \quad (w \in H^r(\Gamma, M)).$$

Since Res preserves the cup product, it follows that we can define a ring homomorphism $HH^*(\Gamma) \rightarrow H^*(\Lambda, \Gamma)$. Moreover, using the embedding of Λ -bimodules $\Lambda \rightarrow \Gamma$, we can define

$$HH^*(\Lambda) \rightarrow H^*(\Lambda, \Gamma) \xrightarrow{\text{Cor}} HH^*(\Gamma).$$

In particular, we have $Z\Lambda/N_\Lambda(\Lambda) \rightarrow Z\Gamma/N_\Gamma(\Gamma) : \bar{z} \mapsto \overline{N_{\Gamma/\Lambda}(z)}$ in the zero dimension. Note that the zero dimensional complete Hochschild cohomology is different from the ordinary one (cf. [Br]).

On the other hand, using the Λ -bimodule homomorphism $\mu_{\Gamma/\Lambda} : \Gamma \rightarrow \Lambda$, we have

$$HH^*(\Gamma) \xrightarrow{\text{Res}} H^*(\Lambda, \Gamma) \xrightarrow{\mu_{\Gamma/\Lambda}} HH^*(\Lambda).$$

In particular, we have $Z\Gamma/N_\Gamma(\Gamma) \rightarrow Z\Lambda/N_\Lambda(\Lambda) : \bar{z} \mapsto \overline{\mu_{\Gamma/\Lambda}(z)}$ in the zero dimension.

We have some explicit calculations of Res and Cor for twisted group algebras and crossed products (see [S1] and [S2] for twisted group algebras, and [S4] for crossed products).

References

- [Br] M. Broué, On Representations of Symmetric Algebras: An Introduction, *Notes by M. Stricker, Mathematik Department ETH Zürich* (1991)
- [Na] T. Nakayama, On the complete cohomology theory of Frobenius algebras, *Osaka Math. J.* **9** (1957), 165–187
- [No1] T. Nozawa, On the complete relative cohomology of Frobenius extensions, *Tsukuba J. Math.* **17** (1993), 99–113
- [No2] T. Nozawa, On the complete relative homology and cohomology of Frobenius extensions, *Tsukuba J. Math.* **19** (1995), 57–78
- [S1] K. Sanada, On the cohomology of twisted group algebras, *SUT Journal of Math.* **25** (1989), 1–10
- [S2] K. Sanada, On the periodic cohomology of a twisted group algebra, *SUT Journal of Math.* **26** (1990), 1–10
- [S3] K. Sanada, On the cohomology of Frobenius algebras, *J. Pure Appl. Algebra* **80** (1992), 65–88
- [S4] K. Sanada, On the cohomology of Frobenius algebras II, *J. Pure Appl. Algebra* **80** (1992), 89–106

E-mail : sanada@rs.kagu.tus.ac.jp